It was stated earlier that the present method is particularly suited for $n = 2$. The reason is that $I_u(2)$ and $I_{ur}(2)$ are equal whenever $\theta_m + \theta_v - \theta_i$ is an odd-integer multiple of $\pi/2$, as are $I_v(2)$ and $I_{vr}(2)$ and $I_w(2)$ and $I_{wr}(2)$. The semiasymptotic approximations are then, in some sense, a mean of the actual oscillating integrands. They lead to remainders which have the advantage of being simpler than those in Refs. 1-3 but have the disadvantage of being useful over a smaller range of vortex pitch.

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Higher Order Sensitivity Analysis of Complex, Coupled Systems

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Introduction

IN design of engineering systems, the "what if" questions often arise such as, What will be the change of the aircraft payload if the wing aspect ratio is incremented by 10%? Answers to such questions are commonly sought by incrementing the pertinent variable and re-evaluating the major disciplinary analyses involved. These analyses are contributed by engineering disciplines that are usually coupled, as are the aerodynamics, structures, and performance in the context of the preceding question.

The "what if" questions may be answered to the linear order of approximation without using the "increment-and-reevaluate" approach and without finite differencing of the system analysis, by using a method introduced in Ref. 1 for calculating the first derivatives of behavior of coupled systems with respect to design variables. The method called the system sensitivity analysis has been demonstrated in formal optimization that used the derivatives to guide search in multidimensional design space, e.g., Refs. 2 and 3. Obviously, if the problem is strongly nonlinear, the efficiency of such search will improve if second- and, possibly, higher-order derivatives are available to the search algorithm. A need for the second-order derivatives to enhance the method of Ref. 1 applied in such problems is discussed in Ref. 4.

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This Note extends the algorithm of Ref. 1 to include the derivatives of the second and higher orders, again, without finite differencing of the system analysis. It achieves that by recursive application of the same implicit function theorem that underlies Ref. 1. As a supplement to Ref. 1, the Note is not self contained; it references equations in Ref. 1 (such references are shown as "Ref. 1/Eq. No.") and requires that reference as a prerequisite.

First-Order Sensitivity Analysis

The sensitivity problem stated in Ref. 1 calls for calculation of the derivatives of a vector Y solving the governing equations, Ref. 1/Eq. (1), with respect to a design variable X_k. The algorithm developed in Ref. 1 yields the derivative of Y as a solution vector Z of the sensitivity equations, which are linear, simultaneous, algebraic equations of the form

$$AZ = R \tag{1}$$

In Ref. 1, the terms in the preceding equation are defined by two equivalent sets of equations: either Ref. 1/Eq. (4) (based on the residuals and called the Global Sensitivity Equations 1, GSE1) or Ref. 1/Eq. (8) (based on the output/input partial sensitivity derivatives and called the Global Sensitivity Equations 2, GSE2). The contents of the matrix of coefficients A and the right-hand-side vector $\boldsymbol{R}$ are different in the preceding two alternative formulations, but the solution vector Z is the same and represents the derivative of Y with respect to the kth element of the vector of design variables X. It has the meaning of the total derivative because it reflects both the direct and indirect influences of X_k on Y.

Sensitivity Analysis of Second and Higher Orders

Generalization of the preceding first-order sensitivity analysis to higher orders is straightforward by taking advantage of the linearity of Eq. (1) herein. Although most of the linear algebra texts ignore the matter, algorithms for sensitivity analysis of the linear algebraic equations solution have been developed in structural sensitivity analysis. They stem from the same implicit function theorem that was the basis for Ref. 1, e.g., Refs. 5 and 6, and they extend to the second-order derivatives, e.g., Refs. 7 and 8. The pattern established in these algorithms will be adapted to solve the problem at hand.

A compact notation for the derivatives will be used in the remainder of the Note:

$$(\quad)^q_{klm\cdots} = \partial^q(\quad)\,\partial X_k\,\partial X_l\,\partial X_m\cdots \tag{2}$$

where any subscript may by repeated as required to form a high-order, mixed derivative with respect to any combination of variables X.

In the preceding notation, the correspondence of Eq. (1) above to Ref. 1/Eq. (4) and Ref. 1/Eq. (8) makes the derivatives of Z with respect to X equivalent to the derivatives of Y with respect to X, as follows (Z is already the first derivative of Y):

$$\begin{aligned} \boldsymbol{Z}^0 &= \boldsymbol{Y}^1{}_k \\ \boldsymbol{Z}^1{}_l &= \boldsymbol{Y}^2{}_{kl} \\ \boldsymbol{Z}^2{}_{lm} &= \boldsymbol{Y}^3{}_{klm} \\ &\vdots \\ \boldsymbol{Z}^N_{lm\cdots} &= \boldsymbol{Y}^{N+1}{}_{klm}\cdots \end{aligned} \tag{3}$$

Repeated differentiation of Eq. (1) yields the derivatives of Z according to a regular pattern, shown up to the fourth derivative as

$$AZ^1_l = R^1_l - A^1_l Z^0 \tag{4}$$

$$AZ^2_{lm} = R^2_{lm} - A^1_m Z^1_l - D^1_m (A^1_l Z^0) \tag{5}$$

$$AZ^3_{lmn} = R^3_{lmn} - A^1_n Z^2_{lm} - D^1_n (A^1_m Z^1_l) - D^2_{mn} (A^1_l Z^0) \tag{6}$$

$$AZ^4_{lmnp} = R^4_{lmnp} - A^1_p Z^3_{lmn} - D^1_p(A^1_n\ Z^2_{lm})$$

$$-D^2_{np}(A^1_m Z^1_l) - D^3_{mnp}(A^1_l Z^0), \text{ etc.} \tag{7}$$

where D^q_{lmn} ( ) is a shorthand for the qth mixed derivative of the product of the pair of functions named in the parentheses, obtained by the usual rules of the product differentiation. Here, again, any subscript may be repeated. Once the derivatives of Z are obtained, the derivatives of Y are available from Eq. (3).

Special Case of Single Variable

The recursivity of Eqs. (4–7) allows simplification of the algorithm if there is only one variable with respect to which the derivatives are to be taken. It is the case of interest, because it occurs in searching design space along a line defined by a search direction S—an operation that is a part of many optimization algorithms. Accurate extrapolation based on higher-order derivatives may reduce the need for costly repetitions of the system analysis in that operation by widening the move limits. Assuming that the direction vector S has already been generated, all of the variables X_k become linked to the step length h through S so that

$$X_{\text{new}} = X_{\text{old}} + h\,S \tag{8}$$

This relation enables one to substitute X with h in Ref. (1)/Eq. (1), and by chain differentiation to replace the right-hand sides of Ref. 1/Eq. (4) and Ref. 1/Eq. (8) with

$$\text{RHS}(h) = \sum_{k=1}^{d} S_k\ \text{RHS}(X_k) \tag{9}$$

where RHS (h) is the right-hand-side vector reflecting the presence of only a single variable, h; RHS (X_k) refers to the RHS in Ref. 1/Eq. (4), or Ref. 1/Eq. (8), related to a variable X_k; and S_k is the kth element of S, and d is the number of variables X_k.

With the problem reduced to a single variable, and d is the number of varibles h, the variable identification subscripts are no longer needed, and R in Eq. (1) may be replaced with

$$R = \text{RHS}(h) \tag{10}$$

from Eq. (9) to transform the pattern of Eqs. (4–7) to

$$AZ^1 = R^1 - A^1\ Z^0 \tag{11}$$

$$AZ^2 = R^2 - A^1\ Z^1 - D^1(A^1\ Z^0) \tag{12}$$

$$AZ^3 = R^3 - A^1\ Z^2 - D^1(A^1\ Z^1) - D^2(A^1\ Z^0) \tag{13}$$

$$AZ^4 = R^4 - A^1\ Z^3 - D^1(A^1 Z^2) - D^2(A^1 Z^1) - D^3(A^1\ Z^0) \tag{14}$$

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$$AZ^N = R^N - A^1 Z^{N-1} - \sum_{S=1}^{N-1} D_S \times (A^1 Z^{N-1-s-1}) \tag{15}$$

By Leibniz's rule (e.g., Ref. 9), the D terms may be written as

$$D^s(A^1\ Z^{N-1-s-1}) = \sum_{q=0}^{S} (C^s_q\, A^{1+s-q} Z^{N-s-1+q}) \tag{16}$$

where C^q_s are the binomial coefficients

$$C^s_q = s!/[q!(s-q!)] \tag{17a}$$

$$C^s_s = C^s_o = 1 \tag{17b}$$

$$C^s_1 = s \tag{17c}$$

also obtainable from the Pascal's triangle.[9]

Structural Sensitivity Analysis: Special Case of a Linear System

Another special case arises when the governing equations, Ref. 1/Eq. (1), are linear algebraic simultaneous equations. For instance, consider the load-deflection equations in the displacement method of structural analysis

$$KU-P \tag{18}$$

where the stiffness matrix K and the load vector P may be functions of design variables X, so that the vector of displacements becomes an implicit function of X. If one substitutes the terms in Eq. (1) as

$$A = K \tag{19a}$$

$$Z = \frac{\partial U}{\partial X_k} \tag{19b}$$

$$R = \frac{\partial P}{\partial X_k U} - \frac{\partial K}{\partial X_k U} \tag{19c}$$

then Eqs. (3–17) formulate the structural sensitivity analysis through the Nth-order derivatives (including the case of derivatives taken in a direction S in the design space).

For a broad class of design variables, e.g., the wall thickness in a membrane structure, or the moment of inertia of a beam cross section, K is a linear function. Then, the patterns of Eqs. (4–7) and (11–15) simplify drastically because the terms comprising the second and higher derivatives of K vanish (unfortunately, this is not so in overall shape optimization). When one wishes to compute derivatives for only a subset of U, then further simplifications and computational cost savings are available by the use of an alternative known as the adjoint variable method (Refs. 6 and 7 discuss that method including the second derivative applications); that is, however, beyond the scope of this Note.

Computational Cost

The algorithm computational cost for a full set of mixed derivatives of Nth order escalates rapidly with the value of N. First, because the number of derivatives in a complete set is proportional to the number of design variables d raised to the Nth power, and, second, because Eqs. (4–7) are recursive. Inspection of the A and D terms in these equations reveals that, in the calculation of a mixed derivative of Nth order, the recursivity requires derivatives of the orders $N-1$ and lower as prerequisites. For example, to solve Eq. (7) for Z^4_{lmnp}, the prerequisites are the derivatives Z^3_{lmn}, Z^3_{lmp}, Z^3_{lnp}, and Z^3_{mnp}. However, this recursive combinatorial escalation and the associated cost do not occur in the case of a single variable, as shown in Eqs. (8–17).

The cost escalation is moderated somewhat by all of Eqs. (4–7) sharing the same matrix A. The matrix may be factored only once and reused when the equations are solved one after another. Each solution requires, then, only a backsubstitution of its right-hand side over the factored A—a task ideally suited to the modern vector and parallel processor computers. The computational cost of Eqs. (4–7) is the upper bound, and the cases of the single variable [Eqs. (8–17)] and structural sensitivity [Eqs. (18) and (19)] are below that bound owing to the absence of mixed derivatives and the vanishing higher-order derivatives of K, respectively.

In optimization, the cost of the higher-order derivatives is relative to the aggregate cost of analysis and the first-order sensitivity analysis repeated as many times as necessary to produce the equivalent information by successive linearizations within move limits. That relative cost is problem dependent, and it is also strongly influenced by implementation manner of the analysis itself and of the sensitivity algorithms discussed earlier. It appears to be high and the extent to which it might be justified by the higher-order information benefits will have to be learned by applications.

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